Option Price Bounds

1. Stochastic Dominance

Stochastic dominance is a term which refers to a set of relations that may hold between a pair of distributions. A very common application of stochastic dominance is to the analysis of income distributions and income inequality. The concept can, however, be applied in many other domains, in particular financial economics, where the distributions considered are usually those of the random returns to various financial assets. In what follows, there are often clear analogies between things expressed in terms of income distributions and financial counterparts.

In order to determine whether a relation of stochastic dominance holds between two distributions, the distributions are first characterized by their **cumulative distribution functions**, or CDFs. For a given set of incomes, the value of the CDF at income y is the proportion of incomes in the set that are no greater than y. In the context of a random variable Y, the value of the CDF of the distribution of Y at y is the probability that Y should be no greater than y.

Suppose that we consider two distributions F and G, characterized respectively by CDFs F and G. Then distribution F dominates distribution G stochastically at first order if, for any argument y, $G(y) \geq F(y)$. This definition often looks as though it is the wrong way round, but a moment's reflection shows that it is correct as stated. If y denotes an income level, then the inequality in the definition means that the proportion of individuals in distribution G with incomes no greater than y is no smaller than the proportion of such individuals in distribution F. In other words, there is at least as high a proportion of poor people in G as in F, if poverty means an income smaller than y. If F dominates G at first order, then whatever poverty line we may choose, there is always more poverty in G than in F, which is why we say that G is the dominated distribution.

Higher orders of stochastic dominance can also be defined. To this end, we define repeated integrals of the CDF of each distribution. Formally, we define a sequence of functions by the recursive definition

$$D^1(y) = F(y), \quad D^{s+1}(y) = \int_0^y D^s(z)\,dz, \quad \text{for } s = 1, 2, 3.\dots.$$

Thus the function D^1 is the CDF of the distribution under study, $D^2(y)$ is the integral of D^1 from 0 to y, $D^3(y)$ is the integral of D^2 from 0 to y, and so on. By definition, distribution F dominates G at order s if $D^s_G(y) \geq D^s_F(y)$ for all arguments y. The lower limit of 0 is used for clarity of exposition; in general it is the lowest income in the pooled distributions. The definition makes it clear that first-order dominance implies dominance at all higher orders, and more generally that dominance at order s implies dominance at all orders higher than s. Since the implications go in only one direction, it follows that higher-order dominance is a weaker condition than lower-order dominance.

In theoretical arguments, it is sometimes desirable to distinguish weak from strong stochastic dominance. The above definitions are of weak dominance. For strong dominance, it is required that the inequality should be strict for at least one value of the argument y. In empirical investigations, the distinction is of no interest, since no statistical test can detect a significant difference between weak and strong inequalities. A better definition of strong, or strict, dominance for continuous distributions would be to require the existence of a set of positive Lebesgue measure on which the inequality is strict.

Relation between stochastic dominance and welfare

When studying either income inequality or poverty, one is automatically in a normative context. It can be argued – see Blackorby and Donaldson (1980) – that the social welfare in a given population can be usefully represented by the expected value in the population of some increasing utility function u. If the CDF of income is denoted by F, the measure of welfare is

$$\mathrm{E}_F(u(Y)) = \int_0^\infty u(y)\,\mathrm{d}F(y).$$

Welfare is greater in society F than in society G if $\mathrm{E}_F\big(u(Y)\big) > \mathrm{E}_G\big(u(Y)\big)$, that is, if

$$\int_0^\infty u(y)\,\mathrm{d}(F-G)(y) > 0.$$

On integration by parts, this condition becomes

$$\int_0^\infty (G-F)(y)u'(y)\mathrm{d}y > 0, \tag{1}$$

and so, since $u'(y) > 0$ if u is increasing, welfare in F is greater than in G if $G(y) - F(y) \geq 0$ for all y with strict inequality on some set of positive Lebesgue measure, that is if F stochastically dominates G at first order.

A more restricted class of utility functions requires that the second derivative of u is negative. This corresponds to diminishing marginal utility of income, a property often treated as equivalent to risk aversion. In that case, greater welfare in F than in G holds with a less strict requirement. If we integrate (1) by parts once more, the inequality is

$$-\int_0^\infty (D^2_G - D^2_F)(y)\,u''(y)\,\mathrm{d}y > 0.$$

If $u''(y) < 0$, this means that second-order stochastic dominance of G by F ensures greater welfare in F according to this looser requirement.

The above analysis shows that stochastic dominance is a sufficient condition for greater welfare in F than in G by the two criteria. It is easy to see that stochastic dominance is also necessary if we let u be an arbitrary increasing function for first order, or an arbitrary increasing concave function for second order.

Graphical representation and quantiles

Consider the setup in Figure 1, where the CDFs of two distributions F and G are plotted. The functions D^2 used for second-order dominance comparisons can be evaluated for a given argument, like z_1 in the figure, as the areas beneath the CDFs, by the usual geometric interpretation of the Riemann integral. We see that distribution F dominates G at second order because, although the CDFs cross, the areas between them are such that the condition for second-order dominance is always satisfied. Thus the vertical line MN marks off a large positive area between the graphs of the two CDFs up to the point at which they cross, and thereafter a small negative area bounded on the right by MN.

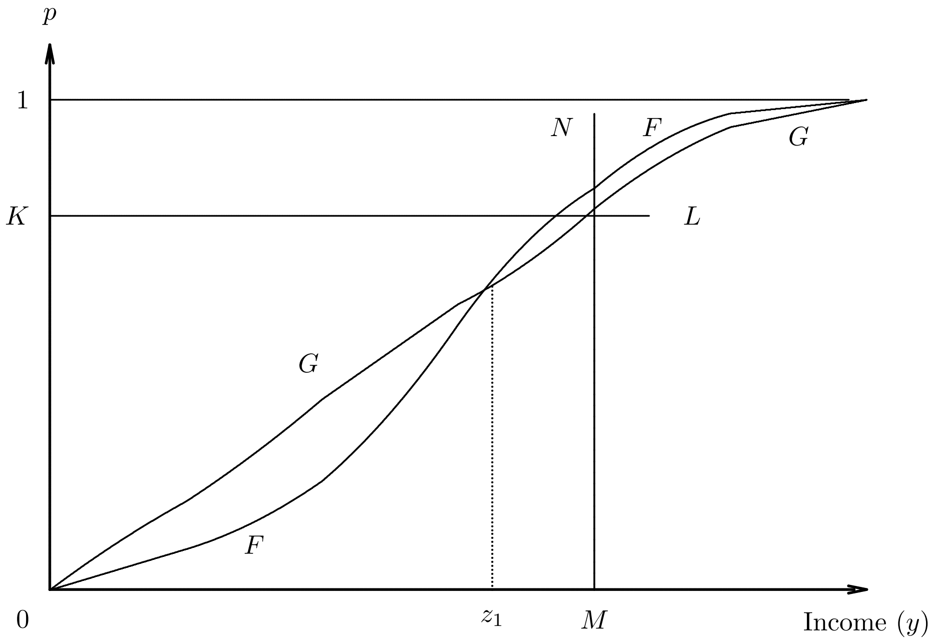

Figure 1. Second Order Dominance

Stochastic Dominance in terms of Quantiles

If F is a CDF defined on the non-negative real line $\mathbb{R}_+$, the corresponding **quantile function** Q maps the interval $[0, 1]$ to $\mathbb{R}_+$. The definition is

$$Q(p) = \inf_{x \in \mathbb{R}_+} \{x \; : \; F(x) > p\}. \tag{2}$$

If F is continuous, Q is simply the inverse function, satisfying

$$F\big(Q(p)\big) = p; \quad \text{and} \quad Q\big(F(x)\big) = x.$$

If F has discontinuities, and is, as usual, a cadlag function, then so is Q with definition (2). However, in what follows almost all distributions will be continuous, the only exception being a discontinuity at the origin.

Given two continuous distributions with CDFs F and G defined on the real line $\mathbb{R}$, F dominates G by first-order stochastic dominance (FSD) if for all $x \in \mathbb{R}$ $F(x) \le G(x)$. Let the quantile functions for the two distributions be Q_F and Q_G respectively.

Lemma 1: F dominates G at first order iff for all $p \in [0, 1]$ $Q_F(p) \ge Q_G(p)$.

Proof:

If for all $x \in \mathbb{R}_+$ $F(x) \le G(x)$, then if $p = F(x)$ or, equivalently $Q_F(p) = x$, we have $p = F\big(Q_F(p)\big) \le G\big(Q_F(p)\big)$. But $p = G\big(Q_G(p)\big)$, and so $G\big(Q_G(p)\big) \le G\big(Q_F(p)\big)$. Since G is an increasing function, this implies that $Q_G(p) \le Q_F(p)$. This shows necessity. For sufficiency, suppose that $Q_G(p) \le Q_F(p)$ for all $p \in [0, 1]$. Let $p = G(x)$. Then $Q_G(p) = x \le Q_F\big(G(x)\big)$. But $x = Q_F\big(F(x)\big)$, and so, since Q_F is increasing, we conclude that $F(x) \le G(x)$. ■

This result is very easy, even obvious. The corresponding result for second-order stochastic dominance (SSD) is less straightforward. The definition of second-order dominance of G by F is that, for all $x \in \mathbb{R}_+$,

$$\int_0^x [G(y) - F(y)]\, \mathrm{d}y \ge 0. \tag{3}$$

The next Lemma establishes the relation between integrals of CDFs and integrals of the corresponding quantile functions.

Lemma 2: For any $x \in \mathbb{R}_+$,

$$\int_0^x F(y)\, \mathrm{d}y = xF(x) - \int_0^{F(x)} Q_F(p)\, \mathrm{d}p.$$

Proof:

Change the integration variable in $\int_0^x F(y)\,\mathrm{d}y$ by $y = Q_F(p)$, so that $\mathrm{d}y = \mathrm{d}Q_F(p)$:

$$\int_0^x F(y)\,\mathrm{d}y = \int_0^{F(x)} p\,\mathrm{d}Q_F(p).$$

Integrate the right-hand side by parts. The integral is

$$\big[p\,Q_F(p)\big]_0^{F(x)} - \int_0^{F(x)} Q_F(p)\,\mathrm{d}p = xF(x) - \int_0^{F(x)} Q_F(p)\,\mathrm{d}p. \quad \blacksquare$$

Remark:

If $x = \infty$, the result of Lemma 2 must be modified as follows:

$$\int_0^\infty \big(F(y) - 1\big)\,\mathrm{d}y = -\int_0^1 Q_F(p)\,\mathrm{d}p.$$

The following theorem provides the expression of second-order stochastic dominance in terms of the quantile functions of the distributions.

Theorem 3 F dominates G at second order iff for all $p \in [0, 1]$,

$$\int_0^p [Q_F(q) - Q_G(q)]\,\mathrm{d}q \geq 0. \tag{4}$$

Proof:

Clearly, first-order dominance implies second-order dominance, and, in that case, Lemma 1 yields the result. We may therefore limit attention to the case in which first-order dominance does not hold. In that case, the graphs of the functions F and G must intersect at least once. Note first that, at any intersection point $\hat{x}$ with $F(\hat{x}) = G(\hat{x}) = \hat{p}$, Lemma 2 shows that

$$\int_0^{\hat{x}} [G(y) - F(y)]\,\mathrm{d}y = \int_0^{\hat{p}} [Q_F(p) - Q_G(p)]\,\mathrm{d}p. \tag{5}$$

It may be that there is an interval $[0, x']$ on which $G(x) = F(x)$. The condition in the statement of the Lemma is trivially satisfied for all x in this interval. Whether or not $x' = 0$, let x_0 be the smallest value of $x > x'$ such that $F(x_0) = G(x_0) \equiv p_0$, and let $p' = F(x') = G(x')$.

For all $x \in [x', x_0]$, we must have $F(x) < G(x)$ for the condition to hold in that interval, and so also $Q_F(p) > Q_G(p)$ for all $p \in [0, p_0]$. The condition (3) for second-order dominance implies that

$$\int_0^{x_0} [G(y) - F(y)]\,\mathrm{d}y = \int_{x'}^{x_0} [G(y) - F(y)]\,\mathrm{d}y > 0. \tag{6}$$

From (5), this inequality is equivalent to

$$\int_{p'}^{p_0} [Q_F(p) - Q_G(p)]\,\mathrm{d}p > 0. \tag{7}$$

But the integrands in (6) and (7) are positive, and so the integrals

$$\int_0^x [G(y) - F(y)]\,\mathrm{d}y \quad \text{and} \quad \int_0^p [Q_F(q) - Q_G(q)]\,\mathrm{d}q$$

are increasing in x and p respectively. Thus, for all $p \in [0, p_0]$ and $x \in [0, x_0]$, (3) and (4) are equivalent.

The rest of the proof proceeds by induction on the number of intersection points of the graphs of F and G. Let x_n be such that $p_n = F(x_n) = G(x_n)$ and that (3) and (4) are equivalent for all $x \in [0, x_n]$. There may exist a point $x' > x_n$ with $F(x) = G(x)$ for all $x \in [x_n, x']$, If so, the equivalence of (3) and (4) holds for all $x \leq x'$. If not, set $x' = x_n$. Let $x_{n+1} > x'$ be the smallest value of x such that $F(x_{n+1}) = G(x_{n+1})$, allowing for the possibility that $x_{n+1} = \infty$, and let $p' = F(x') = G(x')$, $p_{n+1} = F(x_{n+1}) = G(x_{n+1})$. For all $x \in [x', x_{n+1}]$, *either* $G(x) > F(x)$ *or* $G(x) < F(x)$. By an argument like that in the proof of Lemma 1, for all $p \in [p', p_{n+1}]$, in the former case $Q_F(p) > Q_G(p)$, and, in the latter, $Q_F(p) < Q_G(p)$. Consequently, for all $x \in [x', x_{n+1}]$ *either* the integral in (3) is increasing in x and that in (4) is increasing in p *or* the integrals are decreasing in x and p respectively. In both cases, for $x \in [x', x_{n+1}]$, the integral in (3) lies between the values of the integral at $x = x_n$ and at x_{n+1}, while the integral in (4) lies between its values at $p = p'$ and $p = p_{n+1}$. At both intersection points (x_n, p_n) and (x_{n+1}, p_{n+1}) the inequalities (3) and (4) are equivalent, and so they are equivalent for all $x \in [x_n, x_{n+1}]$ and $p \in [p_n, p_{n+1}]$. This completes the induction step.

There may exist an interval $[x_m, \infty)$ for some intersection point (x_m, p_m) with $G(x) = F(x)$ for $x \in [x_m, \infty)$ and so also $Q_F(p) = Q_G(p)$ for $p \in [p_m, 1]$. Clearly inequalities (3) and (4) are equivalent over this range if they are equivalent for $x \leq x_m$ and $p \leq p_m$. This completes the proof. $\blacksquare$

Remarks:

Lemma 1 can be found (without proof) in Levy and Kroll (1978). These authors define first- and second-order stochastic dominance using the sets of utility functions in the section on welfare. Theorem 3 is also given in that paper as their Theorem 5′. They base their proof on a graphical approach, but it is essentially the same proof as given here.

I have wanted a proper proof of Theorem 3 for a long time, and was eager to write it down in this note.

Lemma 4: If the expectation of the distribution with CDF F is finite, it is equal to $\int_0^1 Q(p)\,\mathrm{d}p$, where Q is the quantile function that corresponds to the CDF F.

Proof:

The standard definition of the expectation of F is

$$\mathrm{E}_F = \int_0^\infty x\,\mathrm{d}F(x).$$

Change the integration variable by setting $x = Q(p)$, to obtain

$$\mathrm{E}_F = \int_0^1 Q(p)\,\mathrm{d}\big[F\big(Q(p)\big)\big] = \int_0^1 Q(p)\,\mathrm{d}p,$$

as claimed. ∎

Remark:

If the expectation does not exist, both integrals diverge.

2. Bounds on Call Option Prices Based on Stochastic Dominance

We consider a European call option on an underlying security that we will call the stock. A unit of the stock has a price of S at time $t = 0$, and at the terminal time T its price is denoted S_T. Of course S_T is still an unrealised random variable at time 0. If one unit of account – like a dollar or a euro, or some multiple of these – is invested in the stock, the number of units of the stock purchased is $1/S$. The investor's wealth at time T, the **payoff** of one unit of account in the stock, is S_T/S.

If instead one unit is invested in a call option on the stock with strike price K with price C at $t = 0$, the payoff at time T is $\max\big(0, (S_T - K)/C\big)$.

If the CDF of the random variable S_T is denoted by F, the CDF of the payoff at time T for the investor who purchases the stock is given by

$$F_S(x) = \Pr(S_T/S \le x) = \Pr(S_T \le xS) = F(xS), \tag{8}$$

while for the investor who purchases the call option it is

$$F_C(x) = \Pr\big(\max(0, (S_T - K)/C) \le x\big) = \Pr(S_T \le K + xC) = F(K + xC). \tag{9}$$

Note that the probability that the payoff at $t = T$ is zero is $F(K)$, which is the probability that S_T is less than the strike price K. The quantile functions that correspond to F_S and F_C are respectively

$$Q_S(p) = Q(p)/S \quad \text{and} \quad Q_C(p) = \max\big[0, \big(Q(p) - K\big)/C\big]. \tag{10}$$

The CDFs F_S and F_C are depicted in Figure 2, with the intersection point x_0 as shown.

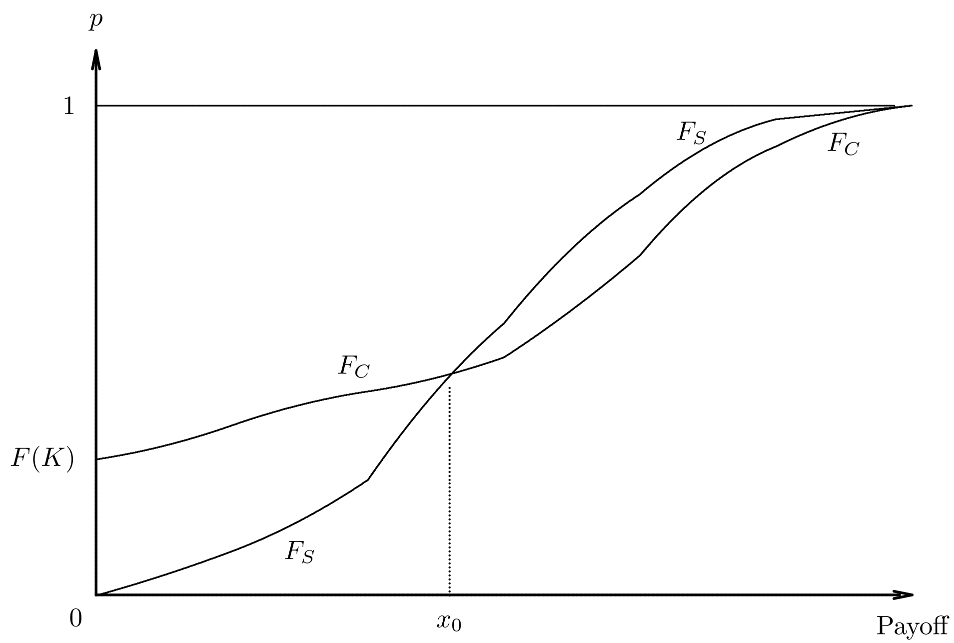

Figure 2. CDFs of payoffs to stock and option

First-order dominance

Suppose that $F_S(x) \le F_C(x)$ for all x with strict inequality for at least one value of x. This means that the probability of ending up with a payoff less than x is smaller with the stock than with the option whatever x may be. No investor with an increasing utility function of wealth would invest in the option in such a case. We call such an investor **rational**. Similarly, if $F_C(x) \le F_S(x)$ for all x no rational investor would purchase the stock. In the first of these circumstances we say that the distribution F_S dominates the distribution F_C stochastically at first order; in the second F_C dominates F_S at first order. If both the stock and the option are traded in equilibrium, we must rule out first-order stochastic dominance in either direction.

This means that the graphs of the CDFs must cross at some point x_0, as seen in Figure 2, at which $F_S(x_0) = F_C(x_0)$, that is $F(x_0 S) = F(K + x_0 C)$, which means that at the intersection point, $x_0 = K/(S - C)$. This implies that x_0 is unique and that $C < S$. If, as we assume, the stock price cannot be negative, $F(0) = 0$ and $F_S(0) = 0$, while $F_C(0) = F(K) > 0$, which means that the graph of F_S cuts that of F_C from below. We see that ruling out first-order dominance of F_C by F_S imposes an upper bound, $C < S$, on values of the call option price C in equilibrium.

For $x < x_0$ $F_S(x) < F_C(x)$; for $x > x_0$ $F_S(x) > F_C(x)$. The probability of a payoff at $t = T$ less than x_0 is the same for the stock and the option, but the probability of getting less than x for some $x < x_0$ is lower for the stock than for the option, thus favouring the stock for a lower value of S_T. Contrariwise, the probability of getting less than $x > x_0$ is greater for the stock than for the option. A high value of S_T thus favours the option.

The bound $C < S$ is neither surprising nor very informative. But it turns out that, if an investor may hold a portfolio that contains a riskless asset as well as the stock or option, both of them risky assets, a portfolio comprised of the option and the riskless asset may dominate or be dominated by a portfolio with the stock and the riskless asset. The rate of return on the riskless asset, which we may imagine to be the interest rate, is denoted by r, so that the payoff at $t = T$ on an investment of one unit of account in the riskless asset is $1 + r$.

Suppose that the one unit is split between a proportion α, $0 \le \alpha \le 1$, invested in an asset yielding a random payoff X at $t = T$ with CDF F, with the remaining proportion $1 - \alpha$ invested in the riskless asset. Then the CDF of the value of the resulting portfolio at $t = T$ is

$$\begin{aligned} F_\alpha(x) &= \Pr\bigl(\alpha X + (1-\alpha)(1+r) \le x\bigr) = \Pr\bigl[X \le \bigl(x - (1-\alpha)(1+r)\bigr)/\alpha\bigr] \\ &= F\bigl[\alpha^{-1}\bigl(x - (1-\alpha)(1+r)\bigr)\bigr]. \end{aligned} \tag{11}$$

If Q_α is the quantile function corresponding to F_α, we have

$$F\bigl(Q(p)\bigr) = p = F_\alpha\bigl(Q_\alpha(p)\bigr) = F\bigl[\alpha^{-1}\bigl(Q_\alpha(p) - (1-\alpha)(1+r)\bigr)\bigr],$$

so that

$$Q_\alpha(p) = \alpha Q(p) + (1-\alpha)(1+r). \tag{12}$$

Lemma 5: If $\alpha \in (0,1]$ then $F_\alpha(1+r) = F(1+r)$.
Further, $F_\alpha(x) < F(x)$ for $x < 1 + r$ and $F_\alpha(x) > F(x)$ for $x > 1 + r$.

Proof:

For the first statement of the Lemma we have

$$F_\alpha(1+r) = F\bigl[\alpha^{-1}\bigl(1+r-(1-\alpha)(1+r)\bigr)\bigr] = F\bigl[\alpha^{-1}\alpha(1+r)\bigr] = F(1+r).$$

For the rest, note that the inequality $\alpha^{-1}\bigl(x - (1-\alpha)(1+r)\bigr) < x$ is equivalent to $x < 1+r$, while $\alpha^{-1}\bigl(x-(1-\alpha)(1+r)\bigr) > x$ is equivalent to $x > 1 + r$. Since F is an increasing function, the result follows. ■

Now consider another risky payoff Y with CDF G. Examples can be constructed where F does not dominate G and is not dominated by it by FSD, but where there exists $\alpha \in [0,1]$ such that F_α dominates G. If F and G are the payoffs from investments in a call option and the underlying stock, such an example is not compatible with market equilibrium, and this fact leads to bounds on the option price. But an investor in Y can also combine the investment with an investment in the riskless asset. If an investment of one unit of account is split in this way, with a proportion of $1 - \beta$ devoted to the riskless asset, then if, for any $\beta \in [0,1]$, there exists α such that F_α dominates G_β, again this is incompatible with equilibrium.

Lemma 6: If F_α dominates G by FSD, then for any β there exists γ such that F_γ dominates G_β.

Proof:

Note first that if F dominates G by FSD, then F_β dominates G_β, for, if $Q_F(p) \ge Q_G(p)$ for all $p \in [0,1]$, then $\beta Q_F(p) + (1-\beta)(1+r) \ge \beta Q_G(p) + (1-\beta)(1+r)$. Thus, since we suppose that F_α dominates G, it follows that $(F_\alpha)_\beta$ dominates G_β. But from (11)

$$\begin{aligned} (F_\alpha)_\beta(x) &= \Pr\bigl[\beta(\alpha X + (1-\alpha)(1+r)) + (1-\beta)(1+r) \le x\bigr] \\ &= \Pr\bigl[\alpha\beta X + [\beta(1-\alpha) + (1-\beta)](1+r) \le x\bigr] \\ &= \Pr\bigl[\alpha\beta X + (1-\alpha\beta)(1+r) \le x\bigr] = F_{\alpha\beta}(x). \end{aligned}$$

The result follows by setting $\gamma = \alpha\beta$. ■

Remark:

Lemma 6 is stated and proved in Levy and Kroll (1978).

As a matter of terminology, if $\{F_\alpha\}$ dominates $\{G_\beta\}$ by FSD, we say that F dominates G by **FSRD**.

Corollary 7: F dominates G by FSD implies that F dominates G by FSRD.

Proof: Set $\alpha = 1$ in Lemma 6.

This result motivates the notation that $\{F_\alpha\}$ dominates $\{G_\beta\}$ if, for any α and β, a value of α can be found such that F_α dominates G_β. Lemma 6 shows that if just one single α can be found such that F_α dominates G, then $\{F_\alpha\}$ dominates $\{G_\beta\}$.

Observe that Lemma 6 imposes no restrictions on either α or β. If $\alpha < 0$ for some portfolio that combines the riskless asset with a risky asset, that asset must be sold short, while, if $\alpha > 1$, some of the riskless asset must be borrowed in order to finance the purchase of the risky asset. In many circumstances either of these two might be possible, but we exclude them from consideration here.

If we try to show that $\{F_\alpha\}$ dominates $\{G_\beta\}$ we have to find α such that F_α dominates G. This implies a search over α, but it can be avoided by checking just one single inequality.

Theorem 8: The condition that $\{F_\alpha\}$ dominates $\{G_\beta\}$ is equivalent to the inequality

$$\inf_{0\le p<F(1+r)} \frac{Q_G(p)-(1+r)}{Q_F(p)-(1+r)} \ge \sup_{F(1+r)<p\le 1} \frac{Q_G(p)-(1+r)}{Q_F(p)-(1+r)} \tag{13}$$

Proof:

By Lemma 6, the dominance relation holds if there exists α such that F_α dominates G, that is, $Q_{F_\alpha}(p) > Q_G(p)$ for all $p \in [0,1]$. By (12), this is equivalent to

$$\begin{aligned} \alpha Q_F(p) + (1-\alpha)(1+r) \ge Q_G(p) \quad &\text{or} \\ \alpha\big(Q_F(p)-(1+r)\big) \ge Q_G(p)-(1+r). & \end{aligned} \tag{14}$$

Consider first the case in which $0 \le p < F(1+r)$, or equivalently $Q_F(p) < 1+r$, so that $Q_F(p)-(1+r)$ is negative. For (14) to hold for all p in this range, we require

$$\alpha < \inf_{0\le p<F(1+r)} \frac{Q_G(p)-(1+r)}{Q_F(p)-(1+r)}.$$

Clearly this condition is equivalent to (14) for all $p \in [0, F(1+r))$.

Next the other case, with $F(1+r) < p \le 1$. Then $Q_F(p) > 1+r$ and the quantity $Q_F(p)-(1+r)$ is positive. For (14) to hold for all p in this range we need

$$\alpha > \sup_{F(1+r)<p\le 1} \frac{Q_G(p)-(1+r)}{Q_F(p)-(1+r)},$$

and so for (14) to hold for all $p \in [0,1]$ it is necessary and sufficient that there exists α such that

$$\inf_{0\le p<F(1+r)} \frac{Q_G(p)-(1+r)}{Q_F(p)-(1+r)} \ge \alpha \ge \sup_{F(1+r)<p\le 1} \frac{Q_G(p)-(1+r)}{Q_F(p)-(1+r)}, \tag{15}$$

When (15) holds, any α that satisfies it is such that F_α dominates G, and, when it does not hold, there is no such α. ■

We now return to the question of potential dominance of the payoff of an investment in a stock by an investment of the same amount in a call option, or *vice versa*. Recall that the CDF of the payoff of an investment of one unit of account in the stock is F_S given by (8), and that of an investment in a call option by F_C given by (9).

Theorem 9: F_C dominates F_S by FSRD if and only if $F_C(1+r) < F_S(1+r)$.

Proof:

The condition is necessary, since by Lemma 5 $(F_C)_\alpha(1+r) = F_C(1+r)$ for any α, so that, if the condition does not hold, there is no α such that $(F_C)_\alpha$ dominates F_S.

For sufficiency, suppose that there exists $\alpha \in [0,1]$ such that $(F_C)_\alpha$ dominates F_S by FSD. From (11), this means that for all $x \ge 0$

$$\begin{aligned} (F_C)_\alpha(x) &= F_C\big[\alpha^{-1}\big(x-(1-\alpha)(1+r)\big)\big] \\ &= F\big[K + C\alpha^{-1}\big(x-(1-\alpha)(1+r)\big)\big] \le F(xS), \end{aligned}$$

and this is equivalent to the condition

$$\alpha K + C\big(x-(1-\alpha)(1+r)\big) < \alpha x S. \tag{16}$$

For $x = 0$, $F_C(0) = F(K) > 0$ while $F_S(0) = 0$ since $S \ge 0$. Thus α must be small enough so that at $x = 0$ $(F_C)_\alpha(0) = 0$. From (16) this means that $\alpha K - C(1-\alpha)(1+r) < 0$, or $\alpha\big(K+(1+r)C\big) < (1+r)C$, so that we have the bound

$$\alpha < \frac{(1+r)C}{K+(1+r)C}. \tag{17}$$

The condition (16) is linear in x, and so it cannot hold for large x if the coefficient of x is positive. Thus (16) can hold only if $C - \alpha S \le 0$, or $\alpha \ge C/S$. For this and (17) to hold simultaneously we must have

$$C/S \le \frac{(1+r)C}{K+(1+r)C}, \quad \text{or} \quad K + (1+r)C \le (1+r)S. \tag{18}$$

Thus (18) is necessary and sufficient for F_C to dominate F_S by FSRD. But $F_C(1+r) = F\big(K+(1+r)C\big)$ and $F_S(1+r) = F\big((1+r)S\big)$, and so if $F_C(1+r) < F_S(1+r)$ we have $K+(1+r)C < (1+r)S$, that is, (18). The proof is complete. ■

Remark:

Theorem 9 is stated in Levy (1985) without proof.

Alternative proof using quantiles:

By Theorem 8, the result of the Theorem holds iff the inequality (13) is satisfied with $Q_G(p) = Q_S(p)$ and $Q_F(p) = Q_C(p)$, with Q_S and Q_C given by (10). We have

$$\frac{Q_S(p) - (1+r)}{Q_C(p) - (1+r)} = \frac{Q(p)/S - (1+r)}{\max\bigl[0, \bigl(Q(p) - K\bigr)/C\bigr] - (1+r)}. \tag{19}$$

Since $F_S(1+r) = F\bigl(1+r)S\bigr)$, and $F_C(1+r) = F\bigl(K+(1+r)C\bigr)$, the condition that $F_C(1+r) < F_S(1+r)$ is equivalent to $K + (1+r)C < (1+r)S$; compare (18).

Consider first p such that $0 \le p \le F_C(1+r)$, which is equivalent to $0 \le Q(p) < K + (1+r)C$, When $Q(p) = 0$, the ratio (19) is equal to 1. For $0 < Q(p) \le K$, it is $1 - Q(p)/\bigl((1+r)S\bigr) < 1$. For $K < Q(p) < K + (1+r)C$, it is

$$\frac{C\bigl((1+r)S - Q(p)\bigr)}{S\bigl(K + (1+r)C - Q(p)\bigr)}. \tag{20}$$

The denominator of this ratio is positive in the range considered, but tends to zero as $p \uparrow F\bigl(K + (1+r)C\bigr)$. The numerator is positive for $Q(p)$ small enough, and tends monotonically to zero as $p \uparrow F\bigl((1+r)S\bigr)$. If $(1+r)S > K + (1+r)C$, it is still positive when the denominator tends to zero, so that the ratio tends to $+\infty$. If the inequality goes the other way, the ratio tends to $-\infty$, and so the left-hand side of the inequality (13) is also $-\infty$. It follows that (13) cannot be satisfied in this case, thereby implying that the inequality $(1+r)S > K + (1+r)C$ is necessary for (13) to hold. If it does hold, the infimum in (13) is the value of the ratio at $p = F(K)$, or $Q(p) = K$, which is, as we saw above, $1 - K/\bigl((1+r)S\bigr)$.

Now consider the right-hand side of (13), with $Q(p) > K + (1+r)C$. The expression (20) can now be rewritten as

$$\frac{C\bigl(Q(p) - (1+r)S\bigr)}{S\bigl(Q(p) - (K + (1+r)C\bigr)}.$$

It is easily seen that the expression is bounded above by C/S. Inequality (13) is thus equivalent to

$$1 - K/\bigl((1+r)S\bigr) > C/S, \quad \text{that is,} \quad (1+r)S > K + (1+r)C, \tag{21}$$

Thus (21) is also sufficient for (13), and this completes the proof. ■

Theorem 10: F_S dominates F_C by FSRD if and only if $C > S$.

Proof:

It is immediate that $C > S$ implies dominance of F_S by F_C, since then for all $x > 0$ $F_S(x) = F(xS) < F(K + xC)$, and the graphs of F_C and F_S do not intersect.

If $C < S$ the graphs do intersect, and they do so only once, at $x = K/(S - C)$, with F_S crossing F_C from below; recall Figure 2. Now $K/(S - C) > 1 + r$, for otherwise the second inequality in (18) would imply dominance of F_S by F_C by FSRD. If there exists $\alpha \in [0, 1]$ such that $(F_S)_\alpha$ dominates F_C, then by Lemma 5 at $x = K/(S - C)$ $(F_S)_\alpha(x) > F_S(x)$. Thus if F_S does not dominate F_C by FSD, neither does $(F_S)_\alpha$. ■

Remarks:

Dominance of F_C by $(F_S)_\alpha$ is possible with $C < S$ if $\alpha > 1$. But we consider this possibility no further. Theorem 10 is stated without proof in Levy (1985).

Theorems 9 and 10 show that in order to avoid first-order dominance in equilibrium,

$$S - K/(1+r) < C < S; \tag{22}$$

see (18) and Theorem 10. If $C < S - K/(1+r)$, the option is too cheap, and no rational investor would consider investing in the stock itself. This bound was derived by Merton (1973).

Second-order dominance

The usual definition of second-order stochastic dominance (SSD) is that F dominates G at second order if, for all $x \in \mathbb{R}_+$,

$$\int_0^x F(y)\,dy \le \int_0^x G(y)\,dy.$$

Theorem 3 shows that this condition is equivalent to

$$\int_0^p Q_F(q\,dq \ge \int_0^p Q_G(q)\,dq.$$

for all $p \in [0, 1]$. Note that FSD implies SSD.

A result for SSD analogous to Lemma 6 is the following:

Lemma 11: If F_α dominates G, then $\{F_\alpha\}$ dominates $\{G_\beta\}$ by SSD.

Proof:

As usual, let Q_F and Q_G denote the quantile functions corresponding to the distributions F and G respectively. Then let Q_{F_α} and Q_{G_β} denote the quantile functions of F_α and G_β. We have

$$Q_{F_\alpha}(p) - Q_G(p) = \alpha Q_F(p) + (1-\alpha)(1+r) - Q_G(p)$$

so that

$$\begin{aligned}\beta\bigl(Q_{F_\alpha}(p) - Q_G(p)\bigr) &= \alpha\beta Q_F(p) + \beta(1-\alpha)(1+r) - \beta Q_G(p)\\ &= \alpha\beta Q_F(p) + (1-\alpha\beta)(1+r) - \beta Q_G(p) - (1-\beta)(1+r)\\ &= Q_{F_{\alpha\beta}}(p) - Q_{G_\beta}(p). \qquad (23)\end{aligned}$$

If F_α dominates G by SSD, then for all $p \in [0,1]$

$$\beta\int_0^p \bigl(Q_{F_\alpha}(q) - Q_G(q)\bigr)\,\mathrm{d}q \geq 0.$$

But by (23) this implies that

$$\int_0^p \bigl(Q_{F_{\alpha\beta}}(q) - Q_{G_\beta}(q)\bigr)\,\mathrm{d}q \geq 0,$$

and this says that $F_{\alpha\beta}$ dominates G_β by SSD. ■

Remarks:

The relation (23) could be used in an alternative proof of Lemma 6.

If $\{F_\alpha\}$ dominates $\{G_\beta\}$ by SSD, we say that F dominates G by **SSRD**.

As in Corollary 7, we see that dominance by SSD implies dominance by SSRD.

Theorem 12: $\{F_\alpha\}$ dominates $\{G_\beta\}$ by SSD, or F dominates G by SSRD iff

$$\sup_{p_0<p\leq 1}\frac{\displaystyle\int_0^p \bigl(Q_G(q)-(1+r)\bigr)\,\mathrm{d}q}{\displaystyle\int_0^p \bigl(Q_F(q)-(1+r)\bigr)\,\mathrm{d}q} \leq \inf_{0\leq p<p_0}\frac{\displaystyle\int_0^p \bigl(Q_G(q)-(1+r)\bigr)\,\mathrm{d}q}{\displaystyle\int_0^p \bigl(Q_F(q)-(1+r)\bigr)\,\mathrm{d}q} \qquad (24)$$

where $p_0 \neq 0$ solves the equation

$$\int_0^{p_0} \bigl(Q_F(q) - (1+r)\bigr)\,\mathrm{d}q = 0. \qquad (25)$$

Proof:

We must first show that there exists p_0 that satisfies (25). Define the function

$$h(p) = \int_0^p \bigl(Q_F(q) - (1+r)\bigr)\,\mathrm{d}q. \qquad (26)$$

Clearly $h(0) = 0$, and, by Lemma 4, $h(1) = \mathrm{E}_F - (1+r) > 0$ if the asset characterised by the distribution F is not dominated by the risk-free asset by SSD. We see that $h'(p) = Q_F(p) - (1+r)$, so that $h'(0) = -(1+r) < 0$ and $h'(p) > 0$ for all $p > F(1+r)$. Thus h decreases from 0 to the right of $p = 0$ but increases up to a positive value at $p = 1$. Consequently there is a unique $p_0 > F(1+r)$ satisfying (25).

The rest of the proof mirrors that of Theorem 8. We have to show that there exists α such that F_α dominates G if and only if (24) holds. Now F_α dominates G by SSD iff for all $p \in [0,1]$

$$\int_0^p \bigl(\alpha Q_F(q) + (1-\alpha)(1+r) - Q_G(q)\bigr)\,\mathrm{d}q > 0.$$

This relation can be rewritten as

$$\alpha\int_0^p \bigl(Q_F(q) - (1+r)\bigr)\,\mathrm{d}q > \int_0^p \bigl(Q_G(q) - (1+r)\bigr)\,\mathrm{d}q. \qquad (27)$$

Consider first values of p such that $0 \leq p < p_0$. The left-hand side of this inequality is negative for p in this range, and so, for (27) to hold we require

$$\alpha < \inf_{0\leq p<p_0}\frac{\displaystyle\int_0^p \bigl(Q_G(q)-(1+r)\bigr)\,\mathrm{d}q}{\displaystyle\int_0^p \bigl(Q_F(q)-(1+r)\bigr)\,\mathrm{d}q}.$$

Next, look at the case in which $p_0 < p \leq 1$. In this range, the left-hand side of (27) is positive, and so (27) requires

$$\alpha > \sup_{p_0<p\leq 1}\frac{\displaystyle\int_0^p \bigl(Q_G(q)-(1+r)\bigr)\,\mathrm{d}q}{\displaystyle\int_0^p \bigl(Q_F(q)-(1+r)\bigr)\,\mathrm{d}q}.$$

It is possible to find α to satisfy (27) for all $p \in [0,1]$ if and only if (24) holds. This proves the claim of the theorem. ■

Remark:

Theorem 12 is proved in Levy and Kroll (1978) as Theorem 4′.

The next theorem gives a result for SSRD like that for FSRD in Theorem 9.

Theorem 13: F_C dominates F_S by SSRD iff $p_c < p_s$, where p_c and p_s are defined by

$$\int_0^{p_c} \big(Q_C(q) - (1+r)\big)\,\mathrm{d}q = 0 \quad \text{and} \quad \int_0^{p_s} \big(Q_S(q) - (1+r)\big)\,\mathrm{d}q = 0, \quad p_c, p_s \neq 0. \tag{28}$$

Proof:

Since FSRD implies SSRD, we first show that the necessary and sufficient condition for FSRD given in Theorem 9 implies the condition $p_c < p_s$. We use the condition for FSRD in the form (18),

$$(1+r)S > K + (1+r)C \tag{18}$$

and recall from (10) that $Q_C(p) = \max\big[0, \big(Q(p) - K\big)/C\big]$ and $Q_S(p) = Q(p)/S$.

Make definitions akin to definition (26):

$$\begin{aligned} h_c(p) &= \int_0^p \big[\max\big(Q(q) - K, 0\big) - (1+r)C\big]\,\mathrm{d}q \quad \text{and} \\ h_s(p) &= \int_0^p \big(Q(q) - (1+r)S\big)\,\mathrm{d}q. \end{aligned} \tag{29}$$

The definitions (28) become $h_c(p_c) = 0 = h_s(p_s)$.

Note that $h_c(0) = h_s(0) = 0$. Consider first values of h_c and h_s for $p \le F(K)$, that is, $Q(p) \le K$. In this range, we see that $h_c(p) = -p(1+r)C$, while

$$h_s(p) < \int_0^p \big[K - (1+r)S\big]\,\mathrm{d}q = p\big(K - (1+r)S\big),$$

whence by (18) $h_s(p) < h_c(p)$ for $p < F(K)$.

For $p \ge F(K)$ we may write

$$h_c(p) = -\int_0^{F(K)} (1+r)C\,\mathrm{d}q + \int_{F(K)}^p \big[Q(q) - K - (1+r)C\big]\,\mathrm{d}q$$

while

$$h_s(p) = \int_0^{F(K)} \big[Q(q) - (1+r)S\big]\,\mathrm{d}q + \int_{F(K)}^p \big[Q(q) - (1+r)S\big]\,\mathrm{d}q. \tag{30}$$

By the argument for $p \le F(K)$ the first integral in h_s is less than that in h_c. But the integral from $F(K)$ to p in h_s is also less than that in h_c, by (18). Thus for all $p > F(K)$, $h_s(p) < h_c(p)$. In particular, $h_s(p_s) < h_c(p_s)$, and so $h_c(p_s) > 0$. As in the proof of Theorem 12, there exists $p_1 < p_c$ such that $h_c'(p) > 0$ for all $p > p_1$. Thus, since $h_c(p_c) = 0$, $h_c(p_s) > 0$ implies that $p_c < p_s$.

Now consider the case in which (18) does not hold, so that F_C does not dominate F_S by FSRD, but $p_c < p_s$. The first condition means that $(1+r)S < K + (1+r)C$. The necessary and sufficient condition for dominance of F_S by F_C from Theorem 12 for our particular case can be written as

$$\inf_{0 \le p < p_c} \frac{h_s(p)}{h_c(p)} \ge \sup_{p_c < p \le 1} \frac{h_s(p)}{h_c(p)}. \tag{31}$$

Now $h_c(0) = h_s(0) = 0$, and so to obtain the limit of $h_s(p)/h_c(p)$ as $p \to 0$, we apply l'Hôpital's rule. Since $h_c(p) = -(1+r)Cp$ for $p < F(K)$, $h_c'(p) = -(1+r)C$ in a neighbourhood of $p = 0$, so that $h_c'(0) = -(1+r)C$. From (30), we see that $h_s'(0) = Q(0) - (1+r)S = -(1+r)S$, and so we conclude that

$$\lim_{p \to 0} \frac{h_s(p)}{h_c(p)} = S/C > 1.$$

As p increases from 0, both $h_s(p)$ and $h_c(p)$ are negative, as are $h_s'(p)$ and $h_c'(p)$. The ratio $h_s(p)/h_c(p)$ is positive, but decreasing in p near $p = 0$: observe that, for $p \le F(K)$,

$$\frac{h_s(p)}{h_c(p)} = \frac{(1+r)S - \dfrac{1}{p}\displaystyle\int_0^p Q(q)\,\mathrm{d}q}{(1+r)C}, \tag{32}$$

which is decreasing in p from the value S/C at $p = 0$. However, as $p \uparrow p_c$, $h_c(p)$ tends to zero from below. Because $p_s > p_c$, $h_s(p)$ is still negative, and so $h_s(p)/h_c(p) \to +\infty$. It follows that there exists a minimum of the ratio between $p = 0$ and $p = p_c$, at $p = p_0$, say. The first-order condition for the minimum implies that

$$\frac{h_s(p_0)}{h_c(p_0)} = \frac{h_s'(p_0)}{h_c'(p_0)}. \tag{33}$$

Note that it is necessary that $p_0 > F(K)$, since (32) shows that $h_s(p)/h_c(p)$ is decreasing in this range.

Now since $h_s'(p) = Q(p) - (1+r)S$, $h_s'(p) > 0$ for $p > F\big((1+r)S\big)$, and, for $p > F\big(K + (1+r)C\big)$, since $h_c'(p) = Q(p) - K - (1+r)C$, we see that

$h_c'(p) > 0$. At $p = p_c$, $h_c(p)$ switches from negative to positive, and so $h_c'(p_c) > 0$. Consequently, $p_c > F\big(K + (1+r)C\big) > F\big((1+r)S\big)$, and by hypothesis $p_s > p_c$. For $p > p_s$, both $h_s(p)$ and $h_c(p)$ are positive and increasing.

At $p = p_s$ the ratio $h_s(p)/h_c(p)$ is zero, and is positive for all $p > p_s$. As $p \to 1$, there appear to be two possibilities: either the ratio increases for all $p \in [p_s, 1]$, with a positive limiting value at $p = 1$, or it has a maximum somewhere in this range. In the latter case, the first-order condition (33) holds at the maximum. The sign of the derivative of the ratio with respect to p is the sign of $h_c(p)h_s'(p) - h_s(p)h_c'(p)$, and so, in the former case in which this expression is positive for all $p \in [p_s, 1]$, we have

$$\frac{h_s(p)}{h_c(p)} < \frac{h_s'(p)}{h_c'(p)}. \tag{34}$$

At the maximum of the ratio, attained at $p = p_2$, which is either an interior point in the interval $[p_s, 1]$ or else $p_2 = 1$, (34) holds. The inequality (31) will be proved if we can show that the ratio at the minimum p_0 is no smaller than the maximum at p_2, and, by (33) and (34), this holds if $h_s'(p_0)/h_c'(p_0) \geq h_s'(p_2)/h_c'(p_2)$.

For $p = p_0$ or $p = p_2$, we have

$$\frac{h_s'(p)}{h_c'(p)} = \frac{Q(p) - (1+r)S}{Q(p) - \big(K + (1+r)C\big)},$$

and in the interval $[p_0, p_2]$ the derivative of this expression with respect to p has the same sign as

$$\big[Q(p) - \big(K + (1+r)C\big)\big] - \big[Q(p) - (1+r)S\big] < 0,$$

on account of the assumption that $(1+r)S < K + (1+r)C$, used in this part of the proof, which is now complete. ■

Remark:

A somewhat flawed and incomplete proof of Theorem 13 is given in the Appendix of Levy (1985).

The next theorem gives a necessary and sufficient condition for dominance of F_C by F_S by SSRD.

Theorem 14: F_S dominates F_C by SSRD iff the inequality $\mathrm{E}_S > \mathrm{E}_C$ is satisfied.

Proof:

Sufficiency: By Lemma 4 we have

$$\mathrm{E}_S - \mathrm{E}_C = \int_0^1 \big(Q_S(p) - Q_C(p)\big)\,\mathrm{d}p.$$

By Theorem 3 F_S dominates F_C by SSD iff, for all $p \in [0, 1]$,

$$\int_0^p \big(Q_S(p) - Q_C(p)\big)\,\mathrm{d}p \geq 0. \tag{35}$$

Since F_S intersects F_C only once, from below, if (35) holds for $p = 1$, it also holds for all $p \in [0, 1]$, since, as p falls away from 1, the negative part of the area between the graphs of F_S and F_C falls, and finally vanishes after p falls beneath the intersection point: recall Figure 2. Thus $\mathrm{E}_S > \mathrm{E}_C$ implies SSD of F_C by F_S, which trivially also implies dominance by SSRD.

Necessity: Suppose that $\mathrm{E}_S < \mathrm{E}_C$. We show that in this case there cannot exist $\alpha \in [0, 1]$ such that $F_{S\alpha}$ dominates F_C by SSD. Such dominance would imply that

$$\mathrm{E}_C \leq \mathrm{E}_{S\alpha} = \alpha \mathrm{E}_S + (1-\alpha)(1+r) < \mathrm{E}_S,$$

where the last inequality holds because $\mathrm{E}_S > 1 + r$ if the stock is traded in equilibrium. Thus we would have $\mathrm{E}_C < \mathrm{E}_S$, a contradiction.
■

Remark:

Theorem 14 is proved in Levy (1985) in Theorem 5 of that paper.

By ruling out dominance by SSRD between F_C and F_S in either direction, we obtain somewhat tighter bounds on the call price C than those in (22). This is natural, as dominance by SSRD is less constraining than by FSRD, and so requires stricter conditions to be ruled out. The upper bound is a little easier to derive.

Theorem 15: To avoid dominance by SSRD of F_C by F_S, the call price C must be bounded above:

$$C < \frac{S}{\mathrm{E}_F} \int_{F(K)}^1 \big(Q(p) - K\big)\,\mathrm{d}p. \tag{36}$$

This bound is more restrictive than the upper bound in (22).

Proof:

By (10), the difference between $\mathrm{E}_S - \mathrm{E}_C$ is

$$\frac{1}{S}\int_0^1 Q(p)\,\mathrm{d}p - \frac{1}{C}\left[\int_{F(K)}^1 \big(Q(p)-K\big)\,\mathrm{d}p\right],$$

and this difference must be negative to avoid dominance. Since $\mathrm{E}_F = \int_0^1 Q(p)\,\mathrm{d}p$, it is easy to see that this implies the result (36). Further, since

$$\int_{F(K)}^1 \big(Q(p)-K\big)\,\mathrm{d}p < \int_0^1 Q(p)\,\mathrm{d}p = \mathrm{E}_F,$$

the upper bound in (36) is smaller than S, the upper bound in (22).
■

The result about the lower bound is as follows.

Theorem 16:

To avoid dominance by SSRD of F_S by F_C, the call price C must be bounded below:

$$C > S - \frac{K}{1+r} + \frac{1}{p_s(1+r)}\left[KF(K) - \int_0^{F(K)} Q(q)\,\mathrm{d}q\right] \tag{37}$$

where p_s satisfies the equation $h_s(p_s) = 0$. This bound is more restrictive than the lower bound in (22).

Proof:

Since by Theorem 13 F_C dominates F_S by SSRD iff $p_c < p_s$, avoiding this means ensuring that $p_c \geq p_s$, with p_c and p_s defined as before by the equations (29). At the margin, we have $p_s = p_c$, which means that p and C satisfy jointly the two equations $h_c(p_s) = h_s(p_s) = 0$, that is,

$$0 = \int_0^{p_s} Q(q)\,\mathrm{d}q - p_s(1+r)S = -p_s(1+r)C + \int_{F(K)}^{p_s} \big(Q(q)-K\big)\,\mathrm{d}q.$$

The first equation can be solved directly for p_s, since it does not involve the other unknown, C. The second equation then yields

$$p_s(1+r)C = p_s(1+r)S - \int_0^{F(K)} Q(q)\,\mathrm{d}q - K\big(p_s - F(K)\big),$$

which can be solved for C:

$$C = S - \frac{K}{1+r} + \frac{1}{p_s(1+r)}\left[KF(K) - \int_0^{F(K)} Q(q)\,\mathrm{d}q\right].$$

Then $p_c \geq p_s$ if (37) holds.

Since $Q(q) < K$ for $q \in [0, F(K))$, the final term on the right-hand side of (37) is positive, which shows that the bound (37) is more restrictive than the lower bound in (22). ■

Remark:

The bounds in Theorems 15 and 16 are derived in Levy (1985), in a different form and mistakenly using p_c instead of p_s in the bound (37).

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