June 2025
Final Examination

Models for Financial Economics
Economics 765

Take-home exam due before midnight on Thursday June 19th 2025.

This exam comprises 5 pages, including the cover page

1. Let $X \sim \mathrm{N}(\mu, \sigma^2)$. It is known that the moment-generating function of X is

$$\psi(u) = \mathrm{E} \exp uX = \exp\bigl(\mu u + \tfrac{1}{2}\sigma^2 u^2\bigr).$$

Show how to compute the successive moments of X by differentiating the function ψ at $u = 0$.

The *kurtosis* of a random variable is defined to be the ratio of its fourth central moment to the square of its variance. Show that the kurtosis of X is 3.

Let X and Y be independent standard normal random variables. Let θ be a constant, and define random variables

$$V = X \cos\theta + Y \sin\theta \quad \text{and} \quad W = -X \sin\theta + Y \cos\theta.$$

Show that V and W are independent standard normal random variables.

If $W(t)$ is a standard Brownian motion, that is, $W(0) = 0$ and $\mathrm{Var}\, W(1) = 1$, compute $\mathrm{d}W^4(t)$ and then write $W^4(T)$ as the sum of an ordinary Lebesgue integral with respect to time and an Itô integral.

Take expectations of the formula thus obtained for $W^4(T)$, and derive the formula $\mathrm{E}W^4(T) = 3T^2$.

Use the same method to derive a formula for $\mathrm{E}W^6(T)$.

2. Theorem 11.3.2 of Shreve states that, if $Q(t)$ is a compound Poisson process, then, for $0 = t_0 < t_1 < \ldots < t_n$, the increments

$$Q(t_1) - Q(t_0),\ Q(t_2) - Q(t_1),\ \ldots,\ Q(t_n) - Q(t_{n-1}),$$

are independent and time-translation invariant, so that $Q(t_j) - Q(t_{j-1})$ has the same distribution as $Q(t_j - t_{j-1})$. Use this result to show that the compound Poisson process $Q(t)$ is a Markov process. That is, show that, for $0 \le t \le T$ and some (measurable) function $h(x)$, then there is a function $g(t, x)$ such that

$$\mathrm{E}\bigl[h\bigl(Q(T)\bigr) \mid \mathcal{F}(t)\bigr] = g\bigl(t, Q(t)\bigr),$$

where $\mathcal{F}(t)$ is a filtration relative to which $Q(t)$ is adapted.

3. This is Shreve's exercise 4.11. The Black-Scholes-Merton (BSM) price at time t of a European call with expiry time T and strike price K, when the price of the underlying asset is x at time t, is given by the formula

$$c(t, x) = x\, \Phi\bigl(d_+(T - t, x)\bigr) - K\mathrm{e}^{-r(T-t)}\Phi\bigl(d_-(T - t, x)\bigr), \tag{1}$$

where Φ is the standard normal cumulative distribution function, r is the (constant) risk-free interest rate, and

$$d_\pm(\tau, x) = \frac{1}{\sigma_1\sqrt{\tau}} \left[\log\frac{x}{K} + \Bigl(r \pm \tfrac{1}{2}\sigma_1^2\Bigr)\tau\right].$$

This all assumes that the underlying stock price follows a geometric Brownian motion with volatility σ_1. We suppose that (1) is the market price of the option.

Suppose however that the underlying stock price really follows a geometric Brownian motion with a different volatility $\sigma_2 > \sigma_1$. Thus in differential form

$$\mathrm{d}S(t) = \alpha S(t)\,\mathrm{d}t + \sigma_2 S(t)\,\mathrm{d}W(t).$$

We set up a portfolio, the value of which at time t is denoted $X(t)$, with $X(0) = 0$. At each time t, the portfolio is long one European call and short $c_x(t, S(t))$ shares of stock, so that the amount in the money market account, where the interest rate is r, is

$$X(t) - c(t, S(t)) + S(t)c_x(t, S(t)).$$

We remove cash from the portfolio at a rate $\frac{1}{2}(\sigma_2^2 - \sigma_1^2)S^2(t)c_{xx}(t, S(t))$. The differential of the portfolio value is therefore

$$\begin{aligned}\mathrm{d}X(t) = {} & \mathrm{d}c(t, S(t)) - c_x(t, S(t))\,\mathrm{d}S(t) \\ & + r\big[X(t) - c(t, S(t)) + S(t)c_x(t, S(t))\big]\,\mathrm{d}t \\ & - \tfrac{1}{2}(\sigma_2^2 - \sigma_1^2)\,S^2(t)\,c_{xx}(t, S(t))\,\mathrm{d}t\end{aligned}$$

Show that $X(t) = 0$ for all $t \in [0, T]$. Noting that $c_{xx}(t, S(t)) > 0$, show that an arbitrage opportunity exists, in the sense that we can remove cash from the portfolio at a positive rate, and end with zero liability at $t = T$.

4. The lognormal distribution is the distribution of a random variable that is a normal variable exponentiated:

$$X(\mu, \sigma) = \exp(\mu + \sigma Z),$$

where Z is standard normal.

(i) What is the CDF (cumulative distribution function) of the distribution of $X(\mu, \sigma)$?

(ii) What is the expectation of $X(\mu, \sigma)$?

(iii) What conditions on μ and σ are needed to ensure that the distribution of $X(\mu, \sigma)$ dominates that of $X(0, 1)$ by first-order stochastic dominance?

(iv) The distribution of $X(\mu, \sigma)$ may perhaps dominate that of $X(0, 1)$ at second order if the graph of the CDF of $X(\mu, \sigma)$ cuts that of $X(0, 1)$ from below. What conditions on μ and σ ensure this?

5. This is Shreve's exercise 1.15. Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$, and assume that X has a density function $f(x)$ that is positive for all $x \in \mathbb{R}$. Let g be a strictly increasing differentiable function satisfying

$$\lim_{y\to-\infty} g(y) = -\infty, \quad \lim_{y\to\infty} g(y) = \infty,$$

and define the random variable $Y = g(X)$.

Let $h(y)$ be an arbitrary nonnegative function satisfying $\int_{-\infty}^{\infty} h(y)\,\mathrm{d}y = 1$. We want to change the probability measure so that $h(y)$ is the density of the random variable Y. To do this, we define

$$Z = \frac{h\big(g(X)\big)g'(X)}{f(X)}.$$

(i) Show that Z is nonnegative and that $\mathrm{E}Z = 1$. Define $\tilde{P}$ by

$$\tilde{P}(A) = \int_A Z\,\mathrm{d}P \quad \text{for all } A \in \mathcal{F}.$$

(ii) Show that Y indeed has density h under $\tilde{P}$.

6. Let X be a real-valued random variable defined on the probability space $(\Omega, \mathcal{F}, P)$.

(i) Suppose that $\mathrm{E}|X|^p$ exists, where $p > 0$. Prove the *Markov inequality*, which states that, for all $\varepsilon > 0$,

$$P(|X| > \varepsilon) \le \frac{\mathrm{E}|X|^p}{\varepsilon^p}. \tag{2}$$

(ii) Let $g : \mathbb{R} \to \mathbb{R}_+$ be an increasing function. Show that, for all real a,

$$P(X > a) \le \frac{\mathrm{E}(g(X))}{g(a)}. \tag{3}$$

(iii) Suppose that a discrete-time filtration is defined on the measure space $(\Omega, \mathcal{F})$ as a nested set of σ-algebras $\mathcal{F}_t$, $t = 0, 1, 2, \ldots$, with $\mathcal{F}_s \subset \mathcal{F}_t$ for $s < t$. A martingale in this discrete-time context is a stochastic process X_t, defined for $t = 0, 1, \ldots$, and such that X_t is $\mathcal{F}_t$−measurable for all t, and

$$\mathrm{E}\big(X_{t+1} \,|\, \mathcal{F}_t\big) = X_t. \tag{4}$$

Show that (4) implies that, for all $t = 0, 1, \ldots$,

$$\mathrm{E}\big(X_{t+s} \,|\, \mathcal{F}_t\big) = X_t, \quad s = 1, 2, \ldots. \tag{5}$$

(iv) A stopping time τ relative to the filtration $\mathcal{F}_t$ is a random variable that takes on the possible values $0, 1, 2, \ldots, \infty$. It is such that the event $\{\omega : \tau(\omega) = k\}$ belongs to $\mathcal{F}_k$ for all $k = 0, 1, \ldots$. The stopped process $X_{t\wedge\tau}$ is defined by the equation

$$X_{t\wedge\tau}(\omega) = X_{\min(t,\tau(\omega))}(\omega).$$

Show that

$$X_{t\wedge\tau} = \sum_{s=1}^{t-1} \mathrm{I}(\tau = s) X_s + \mathrm{I}(\tau \geq t) X_t.$$

If X_t is a martingale, show that the process $X_{t\wedge\tau}$ is also a martingale.

(v) The σ-algebra $\mathcal{F}_\tau$ defined by a stopping time τ is such that $A \in \mathcal{F}_\tau$ iff the event $A \cap \{\tau = t\} \in \mathcal{F}_t$ for all $t = 0, 1, 2, \ldots$. If $\tau < n$ almost surely for some finite positive integer n, show that, for all $t = 0, 1, \ldots, n$,

$$\mathrm{E}(X_t \,|\, \mathcal{F}_\tau) = \sum_{s=0}^{n} \mathrm{E}\big(X_t \mathrm{I}(\tau = s) \,|\, \mathcal{F}_s\big).$$

7. This exercise is based on Shreve's exercise 7.7, on the Zero-strike Asian call. The payoff at time T of this option is

$$V(T) = \frac{1}{T}\int_0^T S(u)\,\mathrm{d}u$$

Why is it proper to call this option the zero-strike Asian call?

(i) Suppose at time t we have $S(t) = x \geq 0$ and $\int_0^t S(u)\,\mathrm{d}u = y \geq 0$. Use the fact that $\mathrm{e}^{-ru}S(u)$ is a martingale under the risk-neutral measure to compute

$$\mathrm{e}^{-r(T-t)}\widetilde{\mathbb{E}}\left[\frac{1}{T}\int_0^T S(u)\,\mathrm{d}u \;\Big|\; \mathcal{F}(t)\right].$$

Call your answer $v(t, x, y)$.

(ii) Verify that the function $v(t, x, y)$ you obtained in (i) satisfies the Black-Scholes-Merton equation

$$v_t(t, x, y) + rxv_x(t, x, y) + xv_y(t, x, y) + \tfrac{1}{2}\sigma^2 x^2 v_{xx}(t, x, y) = rv(t, x, y)$$

for $0 \leq t < T, x \geq 0, y \in \mathbb{R}$, and the boundary conditions

$$v(t, 0, y) = \mathrm{e}^{-r(T-t)}\Big(\frac{y}{T} - K\Big)_+, \quad 0 \leq t < T, y \in \mathbb{R},$$

$$v(T, x, y) = \Big(\frac{y}{T} - K\Big)_+, \quad x \geq 0, y \in \mathbb{R}.$$

(iii) Determine explicitly the process $\Delta(t) = v_x\big(t, S(t), Y(t)\big)$, and observe that it is not random.

(iv) Use the Itô-Doeblin formula to show that if you begin with initial capital $X(0) = v\big(0, S(0), 0\big)$ and at each time you hold $\Delta(t)$ shares of the underlying asset, investing or borrowing at the interest rate r in order to do this, then at time T the value of your portfolio will be

$$X(T) = \frac{1}{T}\int_0^T S(u)\,\mathrm{d}u.$$