Economics 765

June 10, 2025 **R. Davidson**

Assignment 5

You are asked to do exercises 6.8, 6.9, 11.2, and 11.7 of Volume 2 of Shreve. The essence of these exercises is reproduced below for convenience.

6.8 Consider the SDE

$$\mathrm{d}X(u) = \beta(u, X(u))\,\mathrm{d}u + \gamma(u, X(u))\,\mathrm{d}W(u).$$

We assume that if we begin a process at an arbitrary initial positive value $X(t) = x$ at an arbitrary initial time t, and let it evolve according to the SDE, its value at each time $T > t$ could be any positive number but cannot be nonpositive. For $0 \le t < T$, let $p(t, T, x, y)$ be the transition density for the solution to this equation, by which we mean that the random variable $X(T)$ that evolves from $X(t) = x$ has density $p(t, T, x, y)$ in the y variable. We assume that $p(t, T, x, y) = 0$ for $0 \le t < T$ and $y \le 0$.

Show that $p(t, T, x, y)$ satisfies the *Kolmogorov backward equation*

$$-p_t(t, T, x, y) = \beta(t, x)p_x(t, T, x, y) + \tfrac{1}{2}\gamma^2(t, x)p_{xx}(t, T, x, y).$$

6.9 With the same setup as for the previous exercise, show that $p(t, T, x, y)$ satisfies the *Kolmogorov forward equation*, which we may write as

$$\frac{\partial}{\partial T}p(t, T, x, y) = -\frac{\partial}{\partial y}\big(\beta(T, y)p(t, T, x, y)\big) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\big(\gamma^2(T, y)p(t, T, x, y)\big).$$

In contrast to the Kolmogorov backward equation, in which T and y are held constant and the variables are t and x, here t and x are held constant and the variables are T and y.

Notes on this exercise:

- There is a misprint in Shreve in the forward equation: He writes $\beta(t, y)$ for $\beta(T, y)$.
- In the book, there is a long series of steps that can guide you to the desired result. You are at liberty to make use of this or not as you see fit.
- The partial differential operator in the forward equation is the *adjoint* of that in the backward equation. The latter is of course simpler to use.
- The forward equation is also referred to as the *Fokker-Planck equation*, after two eminent physicists.

11.2 Suppose we have observed a Poisson process up to time s, have see that $N(s) = k$, and are interested in the value of $N(s+t)$ for small positive t. Show that

$$\begin{aligned} \mathrm{P}(N(s+t) = k \mid N(s) = k) &= 1 - \lambda t + O(t^2), \\ \mathrm{P}(N(s+t) = k+1 \mid N(s) = k) &= \lambda t + O(t^2), \\ \mathrm{P}(N(s+t) \geq k+2 \mid N(s) = k) &= O(t^2), \end{aligned}$$

where $O(t^2)$ is used to denote terms involving t^2 or higher powers of t.

11.7 Use Theorem 11.3.2 to prove that a compound Poisson process is Markov. In other words, show that, whenever we are given two times $0 \leq t \leq T$ and a functionm $h(x)$, there is another function $g(t, x)$ such that

$$\mathrm{E}\big[h(Q(T) \mid \mathcal{F}(t)\big] = g(t, Q(t)).$$