Economics 765

June 3, 2025 **R. Davidson**

Assignment 4

You are asked to do exercises 5.5, 5.8, 6.1, and 6.6 of Volume 2 of Shreve. The essence of these exercises is reproduced below for convenience.

5.5 You are asked to prove the following result, Corollary 5.3.2 of Shreve.

Let $W(t)$, $0 \le t \le T$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, and let $\mathcal{F}(t)$, $0 \le t \le T$, be the filtration generated by this Brownian motion. Let $\varTheta(t)$, $0 \le t \le T$, be an adapted process, define

$$Z(t) = \exp\left\{-\int_0^t \varTheta(u)\,\mathrm{d}W(u) - \frac{1}{2}\int_0^t \varTheta^2(u)\,\mathrm{d}u\right\},$$
$$\widetilde{W}(t) = W(t) + \int_0^t \varTheta(u)\,\mathrm{d}u,$$

and assume that $\widetilde{\mathrm{E}}\int_0^T \varTheta^2(u)Z^2(u)\,\mathrm{d}u < \infty$. Set $Z = Z(T)$. Then $\mathrm{E}Z = 1$, and under the probability measure $\widetilde{P}$ defined by

$$\widetilde{P}(A) = \int_A Z(\omega)\,\mathrm{d}P(\omega) \quad \text{for all } A \in \mathcal{F},$$

the process $\widetilde{W}(t)$, $0 \le t \le T$, is a Brownian motion.

Now let $\widetilde{M}(t)$, $0 \le t \le T$, be a martingale under $\widetilde{P}$. Then there is an adapted process $\widetilde{\varGamma}(u)$, $0 \le u \le T$, such that

$$\widetilde{M}(t) = \widetilde{M}(0) + \int_0^t \widetilde{\varGamma}(u)\,\mathrm{d}\widetilde{W}(u), \quad 0 \le t \le T. \tag{1}$$

The suggested steps for the proof are as follows.

(i) Compute the differential of $1/Z(t)$.

(ii) Let $\widetilde{M}(t)$, $0 \le t \le T$, be a martingale under $\widetilde{P}$. Show that $M(t) = Z(t)\widetilde{M}(t)$ is a martingale under P.

(iii) According to Shreve's Theorem 5.3.1 (the one-dimensional martingale representation theorem), there is an adapted process $\varGamma(u)$, $0 \le u \le T$, such that

$$M(t) = M(0) + \int_0^t \varGamma(u)\,\mathrm{d}W(u), \quad 0 \le t \le T.$$

Write $\widetilde{M}(t) = M(t)(1/Z(t))$ and take its differential using Itô's product rule.

(iv) Show that the differential of $\widetilde{M}(t)$ is the sum of an adapted process, which we call $\widetilde{\varGamma}(t)$, times $\mathrm{d}\widetilde{W}(t)$, and zero times $\mathrm{d}t$. Integrate to obtain (1).

5.8 (Usual setup and notation.) Assume that there is a unique risk-neutral measure $\widetilde{P}$, and let $\widetilde{W}(t)$, $0 \le t \le T$, be the Brownian motion under $\widetilde{P}$ obtained by using Girsanov's theorem.

Now let $V(T)$ be an almost surely positive $\mathcal{F}(T)$–measurable random variable (under both of the equivalent measures P and $\widetilde{P}$). According to the risk-neutral pricing formula, the price at time t of a security paying $V(T)$ at time T is

$$V(t) = \widetilde{\mathrm{E}}\Big[V(T)\exp - \int_t^T R(u)\,\mathrm{d}u \;\Big|\; \mathcal{F}(t)\Big], \quad 0 \le t \le T.$$

(i) Show that there exists an adapted process $\widetilde{\Gamma}(t)$, $0 \le t \le T$, such that

$$\mathrm{d}V(t) = R(t)V(t)\,\mathrm{d}t + \frac{\widetilde{\Gamma}(t)}{D(t)}\mathrm{d}\widetilde{W}(t), \quad 0 \le t \le T.$$

(ii) Show that, for each $t \in [0,T]$, the price of the derivative security $V(t)$ at time t is almost surely positive.

(iii) Conclude from (i) and (ii) that there exists an adapted process $\sigma(t)$, $0 \le t \le T$, such that

$$\mathrm{d}V(t) = R(t)V(t)\,\mathrm{d}t + \sigma(t)V(t)\,\mathrm{d}\widetilde{W}(t), \quad 0 \le t \le T.$$

In other words, prior to time T, the price of every asset with almost surely positive price at time T follows a generalised geometric Brownian motion.

6.1 Consider the stochastic differential equation

$$\mathrm{d}X(u) = \big(a(u) + b(u)X(u)\big)\,\mathrm{d}u + \big(\gamma(u) + \sigma(u)X(u)\big)\,\mathrm{d}W(u), \tag{6.2.4}$$

where $W(u)$ is a Brownian motion relative to a filtration $\mathcal{F}(u)$, $u \ge 0$, and we allow $a(u)$, $b(u)$, $\gamma(u)$, and $\sigma(u)$ to be processes adapted to this filtration. Fix an initial time t and an initial position $x \in \mathbb{R}$. Define

$$Z(u) = \exp\left\{\int_t^u \sigma(v)\,\mathrm{d}W(v) + \int_t^u \Big(b(v) - \frac{1}{2}\sigma^2(v)\Big)\,\mathrm{d}v\right\},$$
$$Y(u) = x + \int_t^u \frac{a(v) - \sigma(v)\gamma(v)}{Z(v)}\,\mathrm{d}v + \int_t^u \frac{\gamma(v)}{Z(v)}\,\mathrm{d}W(v).$$

(i) Show that $Z(t) = 1$ and

$$\mathrm{d}Z(u) = b(u)Z(u)\,\mathrm{d}u + \sigma(u)Z(u)\,\mathrm{d}W(u), \quad u \ge t.$$

(ii) By its very definition, $Y(u)$ satisfies $Y(t) = x$ and

$$\mathrm{d}Y(u) = \frac{a(u) - \sigma(u)\gamma(u)}{Z(u)}\,\mathrm{d}u + \frac{\gamma(u)}{Z(u)}\,\mathrm{d}W(u), \quad u \ge t.$$

Show that $X(u) = Y(u)Z(u)$ solves the stochastic differential equation (6.2.4) and satisfies the initial condition $X(t) = x$.

6.6 Problem on the moment-generating function for the Cox-Ingersoll-Ross process.

(i) Let $W_1, \ldots, W_d$ be independent Brownian motions and let a and σ be positive constants. For $j = 1, \ldots, d$, let $X_j(t)$ be the solution of the *Ornstein-Uhlenbeck* stochastic differential equation

$$\mathrm{d}X_j(t) = -\frac{b}{2} X_j(t)\,\mathrm{d}t + \frac{1}{2}\sigma\,\mathrm{d}W_j(t).$$

Show that

$$X_j(t) = \mathrm{e}^{-\frac{1}{2}bt}\left[X_j(0) + \frac{\sigma}{2}\int_0^t \mathrm{e}^{\frac{1}{2}bu}\,\mathrm{d}W_j(u)\right].$$

Show further that, for fixed t, the random variable $X_j(t)$ is normal with

$$\mathrm{E}X_j(t) = \mathrm{e}^{-\frac{1}{2}bt}X_j(0), \quad \mathrm{Var}(X_j(t)) = \frac{\sigma^2}{4b}\Big[1 - \mathrm{e}^{-bt}\Big].$$

(Hint: Use Theorem 4.4.9.)

(ii) Define

$$R(t) = \sum_{j=1}^{d} X_j^2(t), \tag{6.9.18}$$

and show that

$$\mathrm{d}R(t) = \big(a - bR(t)\big)\,\mathrm{d}t + \sigma\sqrt{R(t)}\,\mathrm{d}B(t),$$

where $a = d\sigma^2/4$ and

$$B(t) = \sum_{j=1}^{d}\int_0^t \frac{X_j(s)}{\sqrt{R(s)}}\,\mathrm{d}W_j(s)$$

is a Brownian motion. In other words, $R(t)$ is a Cox-Ingersoll-Ross interest rate process. (Hint: Lévy's theorem to show that $B(t)$ is a Brownian motion.)

(iii) Suppose $R(0) > 0$ given, and define

$$X_j(0) = \sqrt{\frac{R(0)}{d}}.$$

Show then that $X_1(t), \ldots, X_d(t)$ are independent, identically distributed, normal random variables, each having expectation

$$\mu(t) = \mathrm{e}^{-\frac{1}{2}bt}\sqrt{\frac{R(0)}{d}}$$

and variance

$$v(t) = \frac{\sigma^2}{4b}\big[1 - \mathrm{e}^{-bt}\big].$$

(iv) Show that the moment-generating function of the square of a variable X with the $\mathrm{N}(\mu, \sigma^2)$ distribution is

$$\mathrm{E} \exp uX^2 = \frac{1}{\sqrt{1 - 2u\sigma^2}} \exp\left\{\frac{u\mu^2}{1 - 2u\sigma^2}\right\}$$

for $u < 1/2\sigma^2$.

(v) Part (iii) shows that $R(t)$ given by (6.9.18) is the sum of squares of IID normal random variables and hence has a *noncentral* χ^2 *distribution*, the term "noncentral" referring to the fact that $\mu(t) = \mathrm{E}X_j(t)$ is not zero. Show that the moment-generating function of $R(t)$ is

$$\mathrm{E} \exp uR(t) = \left(\frac{1}{1 - 2v(t)u}\right)^{2a/\sigma^2} \exp\left\{\frac{\mathrm{e}^{-bt} uR(0)}{1 - 2v(t)u}\right\}$$

for $u < 1/(2v(t))$.