Economics 765

May 27, 2025 **R. Davidson**

Assignment 3

You are asked to do exercises 3.2, 4.5, 4.19, and 5.11 of Volume 2 of Shreve. The essence of these exercises is reproduced below for convenience.

3.2 Let $W(t)$, $t \geq 0$, be a Brownian motion, and let $\mathcal{F}(t)$, $t \geq 0$, be a filtration for this Brownian motion. Show that $W^2(t) - t$ is a martingale.

4.5 Let $S(t)$ be a positive stochastic process that satisfies the generalised geometric Brownian motion differential equation

$$\mathrm{d}S(t) = \alpha(t)S(t)\,\mathrm{d}t + \sigma(t)S(t)\,\mathrm{d}W(t),$$

where $\alpha(t)$ and $\sigma(t)$ are processes adapted to the filtration $\mathcal{F}(t)$, $t \geq 0$, associated with the Brownian motion $W(t)$, $t \geq 0$.

(i) Make use of the above differential equation and the Itô-Doeblin formula in order to compute $\mathrm{d}\log S(t)$. Simplify so that you have a formula for $\mathrm{d}\log S(t)$ that does not involve $S(t)$.

(ii) Integrate the formula you obtained in (i), and then exponentiate the answer to obtain the solution

$$S(t) = S(0)\exp\left\{\int_0^t \sigma(s)\,\mathrm{d}W(s) + \int_0^t \left(\alpha(s) - \frac{1}{2}\sigma^2(s)\right)\mathrm{d}s\right\}.$$

4.19 Let $W(t)$ be a Brownian motion, and define

$$B(t) = \int_0^t \operatorname{sign}(W(s))\,\mathrm{d}W(s),$$

where

$$\operatorname{sign}(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ -1 & \text{if } x < 0. \end{cases}$$

(i) Show that $B(t)$ is a Brownian motion.

(ii) Use Itô's product rule to compute $\mathrm{d}[B(t)W(t)]$. Integrate both sides of the resulting equation and take expectations. Show that $\mathrm{E}[B(t)W(t)] = 0$ (so that $B(t)$ and $W(t)$ are uncorrelated).

(iii) Verify that

$$\mathrm{d}W^2(t) = 2W(t)\,\mathrm{d}W(t) + \mathrm{d}t.$$

(iv) Use Itô's product rule to compute $\mathrm{d}[B(t)W^2(t)]$. Integrate both sides of the resulting equation and take expectations to conclude that

$$\mathrm{E}[B(t)W^2(t)] \neq \mathrm{E}B(t) \cdot \mathrm{E}W^2(t).$$

Explain why this shows that, although they are uncorrelated normal random variables, $B(t)$ and $W(t)$ are not independent.

5.11 Consider a stock price process whose differential is

$$\mathrm{d}S(t) = \alpha(t)S(t)\,\mathrm{d}t + \sigma(t)S(t)\,\mathrm{d}W(t), \quad 0 \le t \le T.$$

Suppose that an agent must pay a cash flow at rate $C(t)$ at each time t, where $C(t)$, $0 \le t \le T$, is an adapted process (as are $\alpha(t)$, the interest rate process $R(t)$, and $\sigma(t)$). If the agent holds $\Delta(t)$ shares of stock at each time t, then the differential of the portfolio value is

$$\mathrm{d}X(t) = \Delta(t)\,\mathrm{d}S(t) + R(t)\big(X(t) - \Delta(t)S(t)\big)\,\mathrm{d}t - C(t)\,\mathrm{d}t.$$

Show that there is a nonrandom value of $X(0)$ and a portfolio process $\Delta(t)$, $0 \le t \le T$, such that $X(T) = 0$ almost surely. (Hint: Define the risk-neutral measure and apply the result of exercise 5.5 to the process

$$\widetilde{M}(t) = \widetilde{\mathrm{E}}\Big[\int_0^T D(u)C(u)\,\mathrm{d}u \,\Big|\, \mathcal{F}(t)\Big], \quad 0 \le t \le T,$$

where $D(t)$ is the discount process $\exp - \int_0^t R(s)\,\mathrm{d}s$.)