Economics 765

May 20, 2025 **R. Davidson**

Assignment 2

You are asked to do exercises 2.3, 2.9, and 3.7 of Volume 2 of Shreve. The essence of these exercises is reproduced below for convenience.

2.3 Let X and Y be independent standard normal random variables. Let θ be a constant, and define the random variables

$$V = X\cos\theta + Y\sin\theta \quad \text{and} \quad W = -X\sin\theta + Y\cos\theta.$$

Show that V and W are independent standard normal variables.

2.9 Let X be a random variable.

(i) Give an example of a probability space $(\Omega, \mathcal{F}, P)$, a random variable X defined on this probability space, and a function f such that the σ-algebra generated by $f(X)$ is not the trivial σ-algebra $\{\emptyset, \Omega\}$ but is strictly smaller than the σ-algebra generated by X.

(ii) Can the σ-algebra generated by $f(X)$ ever be strictly larger than the σ-algebra generated by X?

3.7 Theorem 3.6.2 provides the so-called *Laplace transform* of the density of the first passage time for Brownian motion (the moment-generating function). Let W be a Brownian motion. Fix $m > 0$ and $\mu \in \mathbb{R}$. For $0 \le t < \infty$, define

$$\begin{aligned} X(t) &= \mu t + W(t), \\ \tau_m &= \min\{t \ge 0; X(t) = m\}. \end{aligned}$$

As usual, we set $\tau_m = \infty$ if $X(t)$ never reaches the level m. Let σ be a positive number and set

$$Z(t) = \exp\left\{\sigma X(t) - \left(\sigma\mu + \frac{1}{2}\sigma^2\right) t\right\}.$$

(i) Show that $Z(t)$, $t \ge 0$, is a martingale.

(ii) Use (i) to conclude that

$$\mathrm{E}\left[\exp\left\{\sigma X(t\wedge\tau_m) - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)(t\wedge\tau_m)\right\}\right] = 1, \quad t \ge 0.$$

(iii) Now suppose $\mu \ge 0$. Show that, for $\sigma > 0$,

$$\mathrm{E}\left[\exp\left\{\sigma m - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)\tau_m\right\} \mathrm{I}(\tau_m < \infty)\right] = 1.$$

Use this fact to show that $P\{\tau_m < \infty\} = 1$ and to obtain the Laplace transform

$$\mathrm{E}e^{-\alpha\tau_m} = e^{m\mu - m\sqrt{2\alpha+\mu^2}} \quad \text{for all } \alpha > 0.$$

(iv) Show that, if $\mu > 0$, then $\mathrm{E}\tau_m < \infty$. Obtain a formula for $\mathrm{E}\tau_m$. (Hint: Differentiate the formula in (ii) with respect to α.)

(v) Now suppose $\mu < 0$. Show that, for $\sigma > -2\mu$,

$$\mathrm{E}\left[\exp\left\{\sigma m - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)\tau_m\right\}\mathrm{I}(\tau_m < \infty)\right] = 1.$$

Use this fact to show that $P\{\tau_m < \infty\} = e^{-2m|\mu|}$ (watch out! there is a misprint here in Shreve, who writes $e^{-2x|\mu|}$), which is strictly less than 1, and to obtain the Laplace transform

$$\mathrm{E}e^{-\alpha\tau_m} = e^{m\mu - m\sqrt{2\alpha+\mu^2}} \quad \text{for all } \alpha > 0.$$