Economics 257D1

December 20xx **R. Davidson**

Christmas Examination

Extracts from McGill regulations for exams:

> All mobile phones, smart phones, smart watches and web-accessible electronic devices must be turned off and must not be in the student's possession during the exam.
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> For examinations requiring the use of a calculator, unless otherwise specified by the examiner, only non-programmable, non-text storing calculators are permitted.

Students are allowed one "cheat sheet", on letter-size paper, handwritten or typed, with any information at all.

Attached are tables of the standard Normal, Student's-t, and χ^2 distributions. If a number you need is not in the tables, interpolate roughly and state where you obtained your numbers. Great accuracy is not expected or needed.

1. In each of the following cases, state whatever you can about the probability of the given random variable, X, lying in the interval $[-3, 3]$. All of the random variables are centred, so that, in each case, $\mathrm{E}(X) = 0$.

a) X has the t_4 distribution.

b) The standard deviation of X is 2; its distribution is approximately symmetric, and its fourth central moment is 25.

c) X is the sample mean of a random sample of 200 observations from a distribution with a standard deviation of 5.

d) You toss a coin 5 times, and X = the number of tails minus the number of heads. Avoid writing out all 2^5 possibilities!

2. You have set up your own insurance company to offer insurance to computer owners. Every year in Montreal 1.5% of computer users have their computers stolen. They have a mean insured value of \$500 per computer, and you decide to set premiums at \$20 per \$1000 of insured value, in the hope of making a small profit. You take on 10,000 customers, and so your income from premiums is $\$\ 10,000 \times 500 \times 20/1000 = \$100,000$.

a) In your first year, 1.5% of your clients report a theft. However, to your surprise, what you have to pay out in claims exceeds 1.5% of insured value, and so you take a loss. What must be true of the distribution of the values of stolen computers as opposed to the overall distribution of computer values? Why might this arise?

b) When making your business plan, you had thought that your total payouts would be a realisation from a Normally distributed random variable with expectation \$75,000 ($0.015 \times \$500 \times 10,000$ customers) and a standard deviation of \$10,000. What did you think the probability of making a loss was, given that you expected \$100,000 in income from premiums?

c) A client who takes careful precautions against theft has a theft probability of 1%, but one who does not do so has a theft probability of 2%. Half of your clients are of each type, but you do not know which is which. A particular client takes out insurance, and after one year reports no theft. What is the probability that this is a client who takes careful precautions to avoid being robbed?

d) Explain in words how you might use statistical measures to adjust each individual's premium each year in order to increase your profits.

3. You have a sample of 225 senior citizens who are eligible for a public plan that pays for prescription drugs. The mean cost of an individual's annual prescription drugs in the sample is \$420, and the standard error of annual prescription drug cost is \$100. The total number of people in the sample who charged at least some drug costs to the public plan was 170.

a) Give 90 and 99 per cent confidence intervals for mean drug cost. Make clear the basis for your answer, where in the tables you found the numerical valuey you used, and the details of your calculations.

b) What is the estimated variance of the *proportion* of seniors who charged some costs to the public plan?

c) Give 90 and 99 percent confidence intervals for this proportion.

d) A politician claims that less than 60% of the population of seniors uses the plan. Evaluate this claim statistically on this sample.

4. A referendum is called on the yes/no question of removing the mayor of a large city. The population is divided. You poll 500 people, and find 270 (that is, 54%) who say that they will vote yes to remove; the remainder will vote no.

a) Given this sample, what is the probability that, in fact, a majority of the population plans to vote no?

b) Given this sample, what is the probability that yes will win by 60% or more?

c) How many people would you need to poll, if the *true* proportion who will vote yes is in fact 55%, to be 95% confident that yes will win the referendum? (*Hint:* Obtain a 95% confidence interval bounded by zero on one side.)

5. Two car sales people, Alice and Bob, have random income from commissions of X and Y respectively. Each of their incomes is highly variable, and they would like to reduce their income risk. Alice's sales are steadier than Bob's, but Bob has some hot streaks. We have $\mathrm{Var}(Y) > \mathrm{Var}(X)$ but the expected incomes $\mathrm{E}(X)$ and $\mathrm{E}(Y)$ are the same.

Bob proposes to Alice that they pool and share their incomes, not necessarily 50–50, but in such a way that each person keeps a proportion α of his/her own income, but gets $1 - \alpha$ of the other's. For instance, Alice might keep 70% of her own income and get 30% of Bob's, and Bob would then keep 70% of his own income and get 30% of Alice's.

Each person wants to reduce the variance of commission income without reducing mean income. The covariance of the two incomes X and Y is zero.

a) Bob proposes $\alpha = 0.5$ to Alice. Under what conditions on $\mathrm{Var}(X)$ and $\mathrm{Var}(Y)$ will this make Alice better off? Show that Bob is always better off with this deal.

b) Find the best value of α for Alice if $\mathrm{Var}(Y) = 2\,\mathrm{Var}(X)$. Explain why this is not $\alpha = 0.5$.

c) If $\mathrm{Var}(Y)$ is much bigger than $\mathrm{Var}(X)$, should Alice just keep her own income? ($\alpha = 1$)? Work out the optimal α as a function of the two variances.

d) After they make a deal, both Alice and Bob experience less stress. A third salesperson, Charlie with random income Z wants to join in. Charlie's mean $\mathrm{E}(Z)$ is the same as $\mathrm{E}(X)$ and $\mathrm{E}(Y)$, but his variance is even higher than Bob's. His income is again independent of the others'. He proposes an equal-sharing deal: pool all income and each take a third. This clearly makes Charlie better off. But if $\mathrm{Var}(Z) = 2\,\mathrm{Var}(Y)$ and $\mathrm{Var}(Y) = 2\,\mathrm{Var}(X)$, would this deal make Alice better off? What about Bob?